Adaptive Control

Chapter 7: Digital Control Strategies

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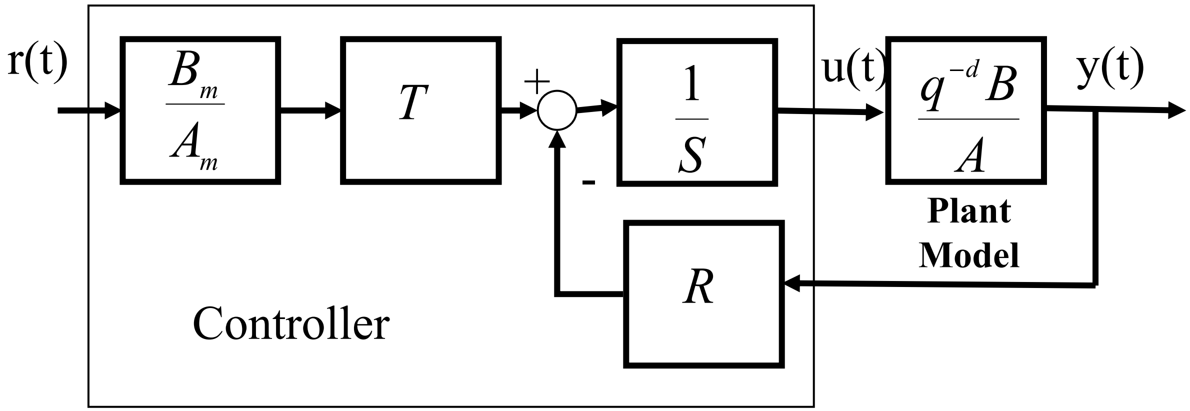
Abstract Building an adaptive control system supposes that for the case of known plant parameters a controller achieving the desired performances can be designed. Therefore this chapter reviews a number of digital control strategies used for the design of the underlying controller whose parameters will be adapted. Pole placement, tracking and regulation with independent objectives , minimum variance control, generalized predictive control, linear quadratic control are presented in detail.

The R-S-T Digital Controller



Plant Model:

$$G(q^{-1}) = H(q^{-1}) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})} = \frac{q^{-d-1}B^*(q^{-1})}{A(q^{-1})}$$

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{n_A} q^{-n_A} \quad B(q^{-1}) = b_1 q^{-1} + \ldots + b_{n_B} q^{-n_B} = q^{-1}B^*(q^{-1})$$

R-S-T Controller:

$$S(q^{-1})u(t) = T(q^{-1})y^*(t+d+1) - R(q^{-1})y(t)$$

Characteristic polynomial (closed loop poles):

$$P(q^{-1}) = A(q^{-1})S(q^{-1}) + q^{-d}B(q^{-1})R(q^{-1})$$

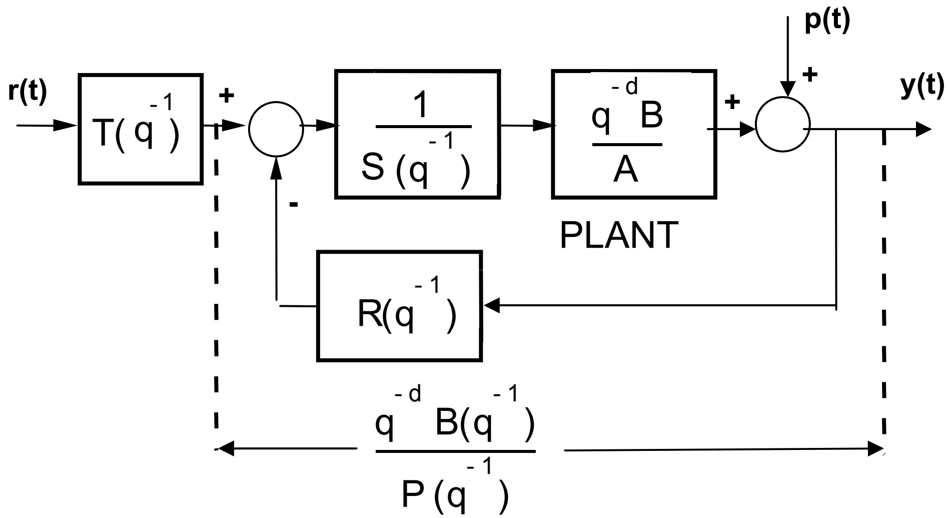
Pole placement

The pole placement allows to design a R-S-T controller for
- stable or unstable systems
- without restriction upon the degrees of A and B polynomials
- without restrictions upon the plant model zeros (stable or unstable)

It is a method that does not simplify the plant model zeros

The digital PID can be designed using pole placement

Structure



Plant: $$H(q^{-1}) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})}$$

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{n_A} q^{-n_A} \qquad B(q^{-1}) = b_1 q^{-1} + b_2 q^{-2} + \ldots + b_{n_B} q^{-n_B} = q^{-1} B^*(q^{-1})$$

Pole placement

Closed loop T.F. (r $\longrightarrow$ y) *(reference tracking)*

$$H_{BF}(q^{-1}) = \frac{q^{-d}T(q^{-1})B(q^{-1})}{A(q^{-1})S(q^{-1}) + q^{-d}B(q^{-1})R(q^{-1})} = \frac{q^{-d}T(q^{-1})B(q^{-1})}{P(q^{-1})}$$

$$P(q^{-1}) = A(q^{-1})S(q^{-1}) + q^{-d}B(q^{-1})R(q^{-1}) = 1 + p_1 q^{-1} + p_2 q^{-2} + \ldots.$$

Defines the (desired)closed loop poles

Closed loop T.F. (p $\rightarrow$ y) *(disturbance rejection)*

$$S_{yp}(q^{-1}) = \frac{A(q^{-1})S(q^{-1})}{A(q^{-1})S(q^{-1}) + q^{-d}B(q^{-1})R(q^{-1})} = \frac{A(q^{-1})S(q^{-1})}{P(q^{-1})}$$

Output sensitivity function

Choice of desired closed loop poles (polynomial *P*)

$$P(q^{-1}) = \underbrace{P_D(q^{-1})}_{\text{Dominant poles}} \underbrace{P_F(q^{-1})}_{\text{Auxiliary poles}}$$

Dominant poles **Auxiliary poles**

Choice of $P_D(q^{-1})$(dominant poles)

Specification in continuous time (t_M, M) $\longrightarrow$ 2nd order (ω_0, ζ) $\xrightarrow[T_e]{\text{discretization}}$ $P_D(q^{-1})$

$$0.25 \le \omega_0 T_e \le 1.5$$
$$0.7 \le \zeta \le 1$$

Auxiliary poles

- *Auxiliary poles are introduced for robustness purposes*
- *They usually are selected to be faster than the dominant poles*

Regulation(computation of *R(q⁻¹)* and *S(q⁻¹)*)

(Bezout) $$A(q^{-1})S(q^{-1}) + q^{-d}B(q^{-1})R(q^{-1}) = P(q^{-1}) \quad \textbf{(*)}$$

? **?**

A and B do not have common factors

$$n_A = \deg A(q^{-1}) \quad n_B = \deg B(q^{-1})$$

unique minimal solution for :

$$n_P = \deg P(q^{-1}) \le n_A + n_B + d - 1$$

$$n_S = \deg S(q^{-1}) = n_B + d - 1 \qquad n_R = \deg R(q^{-1}) = n_A - 1$$

$$S(q^{-1}) = 1 + s_1 q^{-1} + \ldots s_{n_S} q^{-n_S} = 1 + q^{-1} S^*(q^{-1})$$

$$R(q^{-1}) = r_0 + r_1 q^{-1} + \ldots r_{n_R} q^{-n_R}$$

Computation of *R(q-1)* and *S(q-1)*

Equation (*) is written as: $\boxed{Mx = p}$ $\longrightarrow x = M^{-1}p$

$$x^T = [1, s_1, \ldots, s_{n_S}, r_0, \ldots, r_{n_R}] \qquad\qquad p^T = [1, p_1, \ldots, p_i, \ldots, p_{n_P}, 0, \ldots, 0]$$

$$
\begin{array}{c}
\begin{array}{cc}
\overbrace{\hspace{5cm}}^{n_B + d} & \overbrace{\hspace{5cm}}^{n_A}
\end{array}
\\
\underbrace{\begin{bmatrix}
1 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\
a_1 & 1 & & . & b'_1 & & & \\
a_2 & & & 0 & b'_2 & & & b'_1 \\
 & & & 1 & . & & & b'_2 \\
 & & & a_1 & . & & & . \\
a_{n_A} & & & a_2 & b'_{n_B} & & & . \\
0 & & & . & 0 & . & . & . \\
0 & \ldots & 0 & a_{n_A} & 0 & 0 & 0 & b'_{n_B}
\end{bmatrix}}_{n_A + n_B + d}
\left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\\1\\1\end{matrix}}\right\} n_A + n_B + d
\end{array}
$$

$b'_i = 0$ pour $i = 0, 1 \ldots d$; $b'_i = b_{i-d}$ pour $i > d$

Use of WinReg or *bezoutd.sci(.m)* for solving (*)

Structure of $R(q^{-1})$ and $S(q^{-1})$

R and S include pre-specified fixed parts (ex: integrator)

$$R(q^{-1}) = R'(q^{-1})H_R(q^{-1}) \qquad S(q^{-1}) = S'(q^{-1})H_S(q^{-1})$$

H_R, H_S, - pre-specified polynomials

$$R'(q^{-1}) = r'_0 + r'_1 q^{-1} + \ldots r'_{n_{R'}} q^{-n_{R'}} \quad S'(q^{-1}) = 1 + s'_1 q^{-1} + \ldots s'_{n_{S'}} q^{-n_{S'}}$$

- The pre specified filters H_R and H_S will allow to impose certain properties of the closed loop.
- They can influence performance and/or robustness

Fixed parts (H_R, H_S). Examples

Zero steady state error (S_{yp} should be null at certain frequencies)

$$S_{yp}(q^{-1}) = \frac{A(q^{-1})H_S(q^{-1})S'(q^{-1})}{P(q^{-1})}$$

Step disturbance : $H_S(q^{-1}) = 1 - q^{-1}$

Sinusoidal disturbance : $H_S = 1 + \alpha q^{-1} + q^{-2}\ ;\ \alpha = -2\cos\omega T_s$

Signal blocking (S_{up} should be null at certain frequencies)

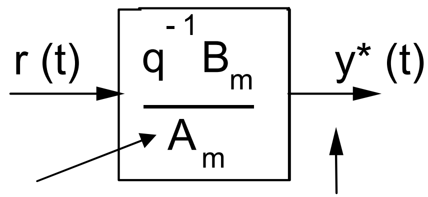
$$S_{up}(q^{-1}) = -\frac{A(q^{-1})H_R(q^{-1})R'(q^{-1})}{P(q^{-1})}$$

Sinusoidal signal: $H_R = 1 + \beta q^{-1} + q^{-2}\ ;\ \beta = -2\cos\omega T_s$

Blocking at 0.5f_{S}: $H_R = (1 + q^{-1})^n\ ;\ n = 1,2$

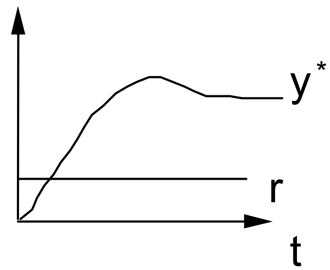
Tracking (computation of $T(q^{-1})$)

Ideal case



Tracking reference model (H_m)

desired trajectory for y (t)



$$H_m(q^{-1}) = \frac{q^{-1}B_m(q^{-1})}{A_m(q^{-1})}$$

$$B_m(q^{-1}) = b_{m0} + b_{m1}q^{-1} + \ldots$$

$$A_m(q^{-1}) = 1 + a_{m1}q^{-1} + a_{m2}q^{-2} + \ldots$$

Specification in continuous time (t_M, M) $\longrightarrow$ 2[nd] order (ω_0, ζ) $\xrightarrow[T_s]{\text{discretization}}$ $H_m(q^{-1})$

$$0.25 \le \omega_0 T_s \le 1.5$$

$$0.7 \le \zeta \le 1$$

The ideal case can not be obtained (delay, plant zeros)

Objective : to approach $y^*(t)$

$$y^*(t) = \frac{q^{-(d+1)}B_m(q^{-1})}{A_m(q^{-1})}r(t)$$

Tracking (computation of $T(q^{-1})$)

Build: $$y^*(t+d+1) = \frac{B_m(q^{-1})}{A_m(q^{-1})} r(t)$$

Choice of $T(q^{-1})$:

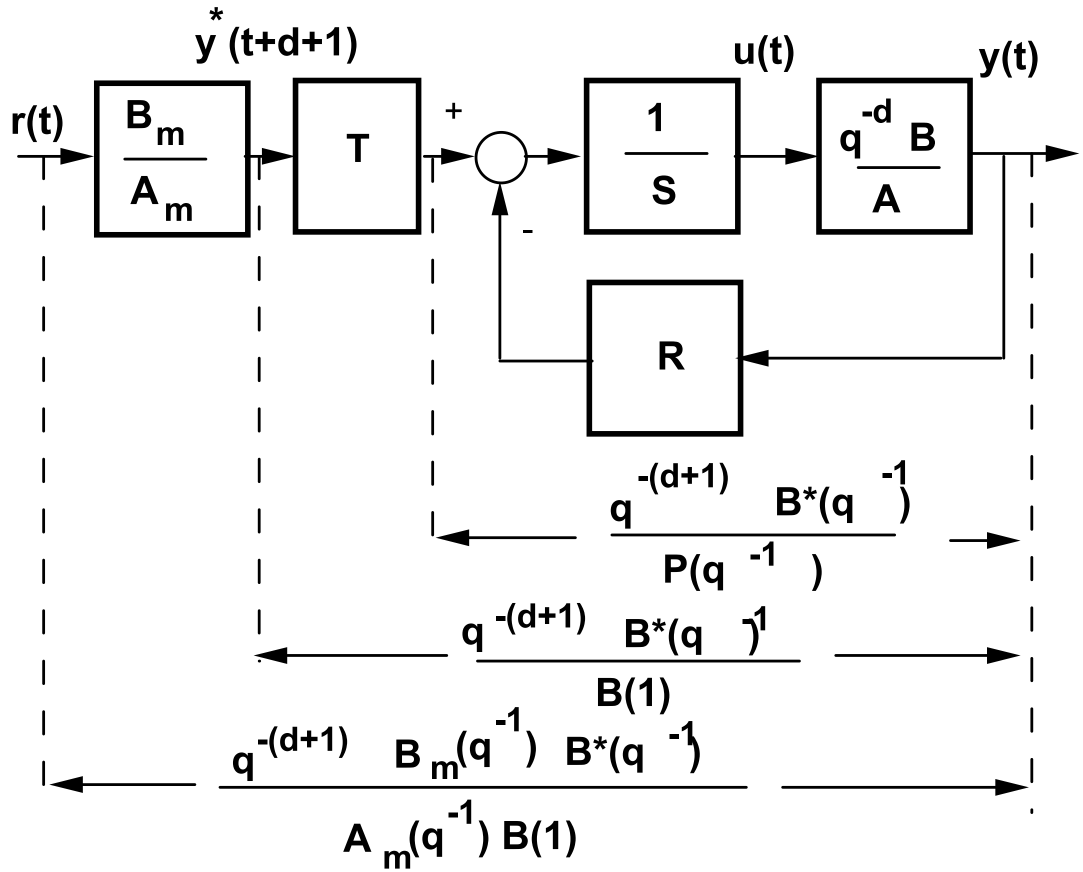
- Imposing unit static gain between y^* and y
- Compensation of regulation dynamics $P(q^{-1})$

$$T(q^{-1}) = GP(q^{-1}) \qquad G = \begin{cases} 1/B(1) & si \quad B(1) \neq 0 \\ 1 & si \quad B(1) = 0 \end{cases}$$

F.T. $r \longrightarrow y$: $$H_{BF}(q^{-1}) = \frac{q^{-(d+1)} B_m(q^{-1})}{A_m(q^{-1})} \cdot \frac{B^*(q^{-1})}{B(1)}$$

Particular case : $P = A_m$ $$T(q^{-1}) = G = \begin{cases} \dfrac{P(1)}{B(1)} & si \quad B(1) \neq 0 \\ 1 & si \quad B(1) = 0 \end{cases}$$

Pole placement. Tracking and regulation



$$S(q^{-1})u(t)+R(q^{-1})y(t)=T(q^{-1})y^*(t+d+1)$$

Pole placement. Control law

$$u(t) = \frac{T(q^{-1})y^*(t+d+1) - R(q^{-1})y(t)}{S(q^{-1})}$$

$$S(q^{-1})u(t) + R(q^{-1})y(t) = GP(q^{-1})y^*(t+d+1) = T(q^{-1})y^*(t+d+1)$$

$$S(q^{-1}) = 1 + q^{-1}S^*(q^{-1})$$

$$\boxed{u(t) = P(q^{-1})Gy^*(t+d+1) - S^*(q^{-1})u(t-1) - R(q^{-1})y(t)}$$

$$y^*(t+d+1) = \frac{B_m(q^{-1})}{A_m(q^{-1})}r(t)$$

$$A_m(q^{-1}) = 1 + q^{-1}A_m^*(q^{-1})$$

$$\boxed{y^*(t+d+1) = -A_m^*(q^{-1})y(t+d) + B_m(q^{-1})r(t)}$$

$$B_m(q^{-1}) = b_{m0} + b_{m1}q^{-1} + \ldots \qquad A_m(q^{-1}) = 1 + a_{m1}q^{-1} + a_{m2}q^{-2} + \ldots$$

Pole placement. Example

Plant : $d=0$

$B(q-1) = 0.1\ q-1 + 0.2\ q-2$

$A(q-1) = 1 - 1.3\ q-1 + 0.42\ q-2$

$Bm(q-1) = 0.0927 + 0.0687\ q-1$

Tracking dynamics →

$Am\ (q-1) = 1 - 1.2451q-1 + 0.4066\ q-2$

$\boldsymbol{Ts = 1s\ ,\ \omega_0 = 0.5\ rad/s,\ \zeta = 0.9}$

Regulation dynamics → $P\ (q-1) = 1 - 1.3741\ q-1 + 0.4867\ q-2$

$\boldsymbol{Ts = 1s\ ,\ \omega_0 = 0.4\ rad/s,\ \zeta = 0.9}$

Pre-specifications : Integrator

***** CONTROL LAW *****

$S\ (q-1)\ u(t) + R\ (q-1)\ y(t) = T\ (q-1)\ y^*(t+d+1)$

$y^*(t+d+1) = [Bm(q-1)/Am(q-1)]\ r(t)$

Controller : $R(q-1) = 3 - 3.94\ q-1 + 1.3141\ q-2$

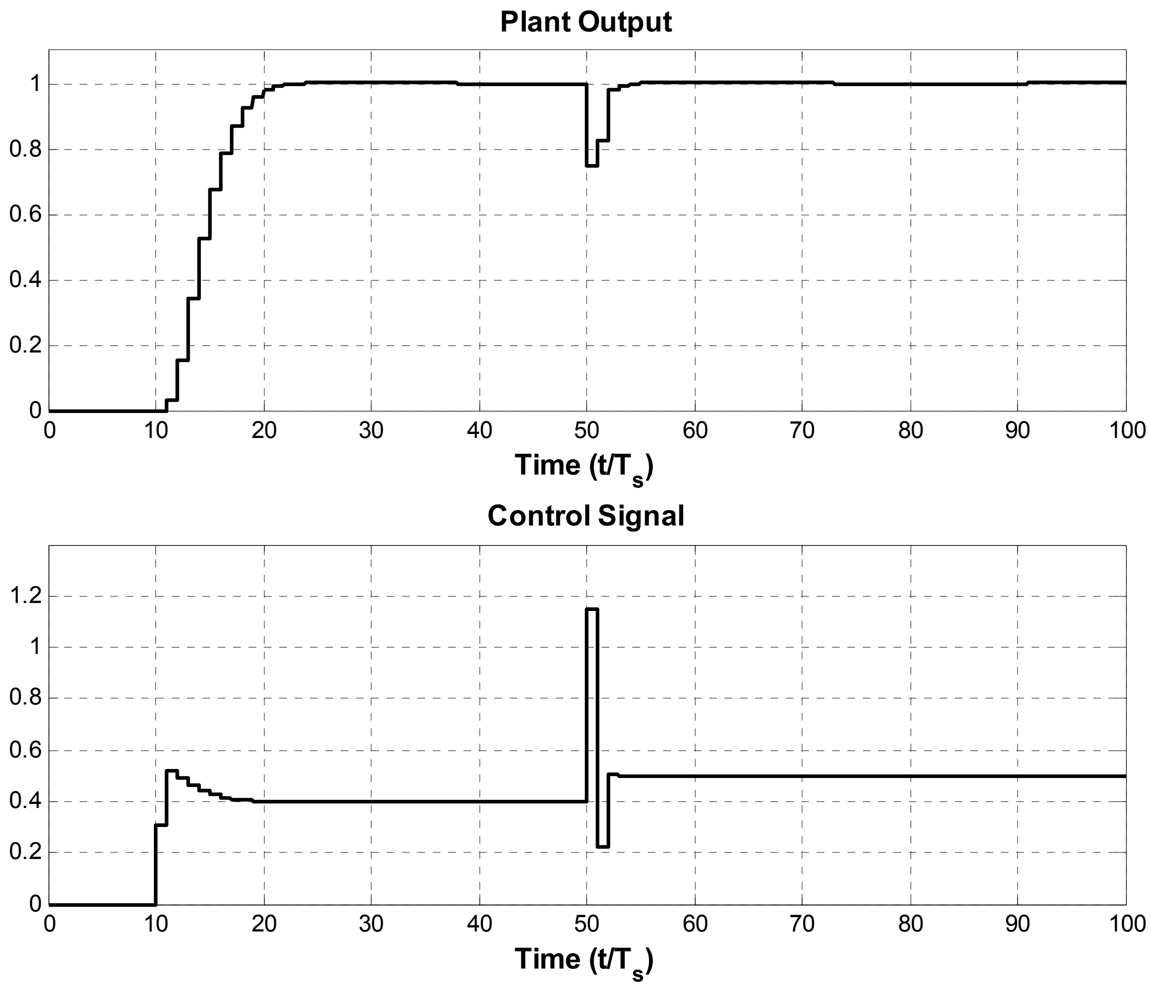
$S(q-1) = 1 - 0.3742\ q-1 - 0.6258\ q-2$

$T(q-1) = 3.333 - 4.5806\ q-1 + 1.6225\ q-2$

Gain margin : 2.703 Phase margin : 65.4 deg

Modulus margin : 0.618 (- 4.19 dB) Delay margin: 2.1. s

Pole placement. Example



Tracking and regulation with independent objectives

It is a particular case of pole placement
(the closed loop poles contain the plant zeros))

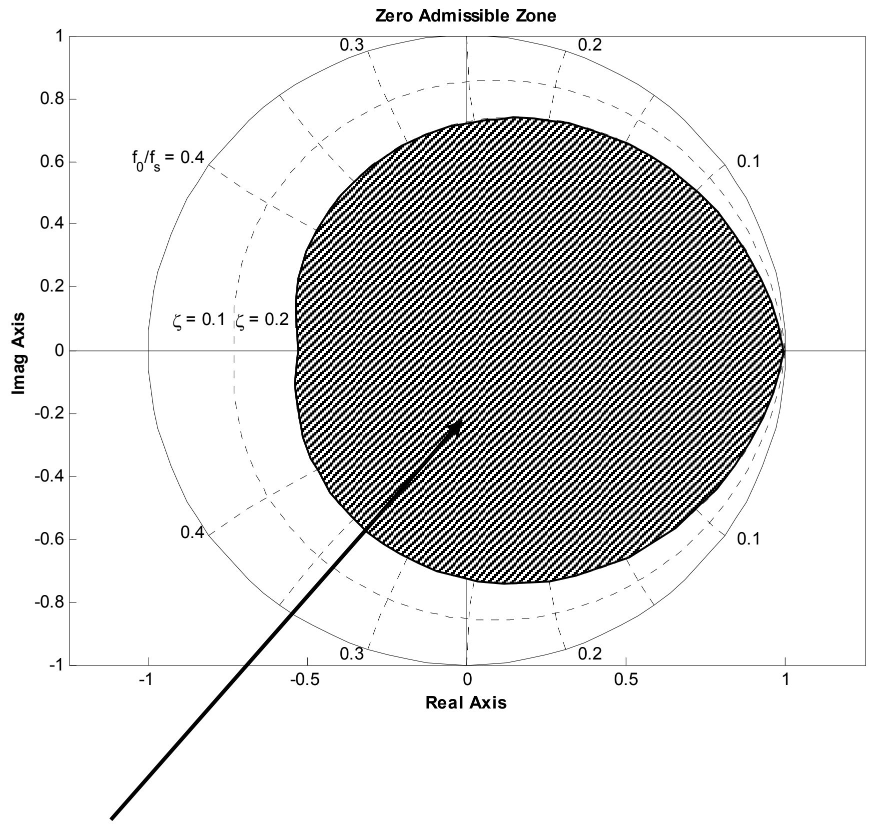
It is a method which simplifies the plant zeros
Allows exact achievement of imposed performances

Allows to design a RST controller for:
- stable or unstable systems
- without restrictions upon the degrees of the polynomials A et B
- without restriction upon the integer delay d of the plant model
- discrete-time plant models with *stable zeros!*

Does not tolerate fractional delay > 0.5 T_S *(unstable zero)*

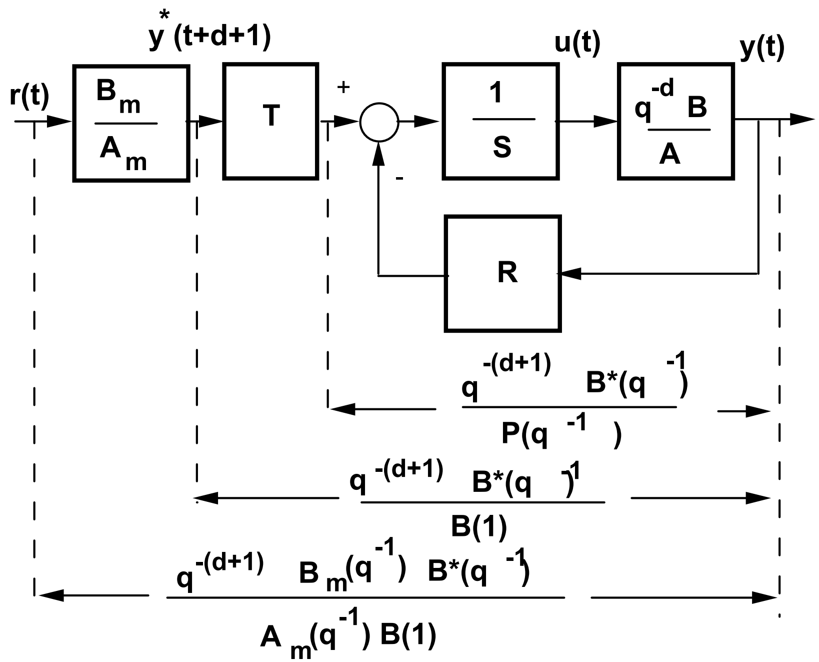
Tracking and regulation with independent objectives

The model zeros should be stable and enough damped



Admissibility domain for the zeros of the discrete time model

Tracking and regulation with independent objectives



$$P(q^{-1}) = P_D(q^{-1})P_F(q^{-1})$$

Reference signal:
(tracking)

$$y^*(t+d+1) = \frac{B_m(q^{-1})}{A_m(q^{-1})} r(t)$$

Regulation (computation of $R(q^{-1})$ and $S(q^{-1})$)

T.F. of the closed loop without *T:*

$$H_{CL}(q^{-1}) = \frac{q^{-d+1}B^*(q^{-1})}{A(q^{-1})S(q^{-1}) + q^{-d+1}B^*(q^{-1})R(q^{-1})} = \frac{q^{-d+1}}{P(q^{-1})} = \frac{q^{-d+1}B^*(q^{-1})}{B^*(q^{-1})P(q^{-1})}$$

The following equation has to be solved :

$$A(q^{-1})S(q^{-1}) + q^{-d+1}B^*(q^{-1})R(q^{-1}) = B^*(q^{-1})P(q^{-1}) \quad (*)$$

S should be in the form: $S(q^{-1}) = s_0 + s_1 q^{-1} + \ldots + s_{n_S} q^{-n_S} = B^*(q^{-1})S'(q^{-1})$

After simplification by B^*,(*) becomes:

$$\boxed{A(q^{-1})S'(q^{-1}) + q^{-d+1}R(q^{-1}) = P(q^{-1})} \qquad (**)$$

Unique solution if: $n_P = deg\, P(q^{-1}) = n_A + d$; $deg\, S'(q^{-1}) = d$; $deg\, R(q^{-1}) = n_A - 1$

$$R(q^{-1}) = r_0 + r_1 q^{-1} + \ldots r_{n_A-1} q^{-n_A-1} \qquad S'(q^{-1}) = 1 + s'_1 q^{-1} + \ldots s'_d\, q^{-d}$$

Regulation (computation of $R(q^{-1})$ and $S(q^{-1})$)

(**) is written as: $Mx = p \longrightarrow x = M^{-1}p$

$$
\begin{array}{c}
n_A + d + 1 \\
\left.\begin{bmatrix}
1 & 0 & & & & & & 0 \\
a_1 & 1 & & & & & & \vdots \\
a_2 & a_1 & & 0 & & & & \vdots \\
\vdots & \vdots & & 1 & & & & \vdots \\
a_d & a_{d-1} & \cdots & a_1 & 1 & & & \vdots \\
a_{d+1} & a_d & & & a_1 & 1 & & \vdots \\
a_{d+2} & a_{d+1} & & & a_2 & 0 & & \vdots \\
 & & & & \vdots & & & 0 \\
0 & 0 & \cdots & 0 & a_{n_A} & 0 & 0 & 1
\end{bmatrix}\right\} n_A + d + 1 \\
\underbrace{\qquad\qquad\qquad\qquad\qquad}_{d+1}\ \underbrace{\qquad\quad}_{n_A}
\end{array}
$$

$$x^T = [1, s'_1, \ldots, s'_d, r_0, r_1, \ldots, r_{n-1}] \qquad p^T = [1, p_1, p_2, \ldots, p_{n_A}, p_{n_A+1}, \ldots, p_{n_A+d}]$$

Use of WinReg or *predisol.sci(.m)* for solving (**)

Insertion of pre specified parts in R and S – same as for pole placement

Tracking (computation of $T(q^{-1})$)

Closed loop T.F.: $r \longrightarrow y$

$$H_{BF}(q^{-1}) = \frac{q^{-(d+1)}B_m(q^{-1})}{A_m(q^{-1})} = \frac{B_m(q^{-1})T(q^{-1})q^{-(d+1)}}{A_m(q^{-1})P(q^{-1})}$$

Desired T.F.

It results : $\boxed{T(q^{-1}) = P(q^{-1})}$

Controller equation:

$$S(q^{-1})u(t) + R(q^{-1})y(t) = P(q^{-1})y^*(t+d+1)$$

$$u(t) = \frac{P(q^{-1})y^*(t+d+1) - R(q^{-1})y(t)}{S(q^{-1})}$$

$$u(t) = \frac{1}{b_1}\left[P(q^{-1})y^*(t+d+1) - S^*(q^{-1})u(t-1) - R(q^{-1})y(t)\right] \qquad (s_0 = b_1)$$

Tracking and regulation with independent objectives. Examples

Plant : $d = 0$

$B(q\text{-}1) = 0.2\ q\text{-}1 + 0.1\ q\text{-}2$

$A(q\text{-}1) = 1 - 1.3\ q\text{-}1 + 0.42\ q\text{-}2$

$\rightarrow Bm\ (q\text{-}1) = 0.0927 + 0.0687\ q\text{-}1$

Tracking dynamics

$\rightarrow Am\ (q\text{-}1) = 1 - 1.2451q\text{-}1 + 0.4066\ q\text{-}2$

$\boldsymbol{Ts = 1s\ ,\ \omega_0 = 0.5\ rad/s,\ \zeta = 0.9}$

Regulation dynamics $\rightarrow P\ (q\text{-}1) = 1 - 1.3741\ q\text{-}1 + 0.4867\ q\text{-}2$

$\boldsymbol{Ts = 1s\ ,\ \omega_0 = 0.4\ rad/s,\ \zeta = 0.9}$

Pre-specifications : Integrator

***** CONTROL LAW *****

$S\ (q\text{-}1)\ u(t) + R\ (q\text{-}1)\ y(t) = T\ (q\text{-}1)\ y^*(t+d+1)$

$y^*(t+d+1) = [Bm\ (q\text{-}1)/Am\ (q\text{-}1)]\ .\ r(t)$

Controller : $R(q\text{-}1) = 0.9258 - 1.2332\ q\text{-}1 + 0.42\ q\text{-}2$

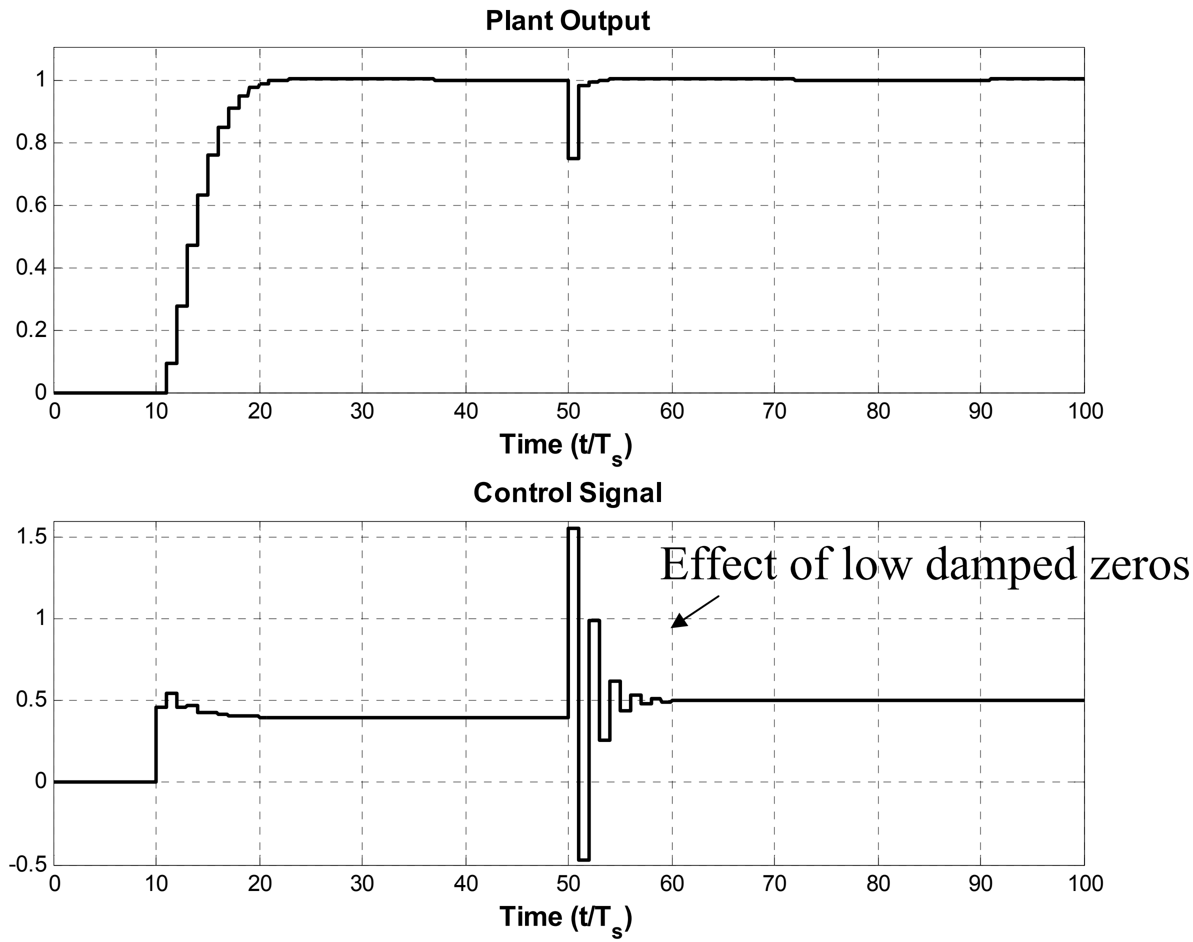
$S(q\text{-}1) = 0.2 - 0.1\ q\text{-}1 - 0.1\ q\text{-}2$

$T(q\text{-}1) = P(q\text{-}1)$

Gain margin : 2.109 Phase margin : 65.3 deg

Modulus margin : 0.526 (- 5.58 dB) Delay margin : 1.2

Tracking and regulation with independent objectives. (d = 0)



The oscillations on the control input when there are low damped zeros can be reduced by introducing auxiliary poles

Internal model control -Tracking and regulation

It is a particular case of the pole placement

The dominant poles are those of the plant model

Does not allow to accelerate the closed loop response

Allows to design a RST controller for:
- well damped stable systems
- without restrictions upon the degrees of the polynomial A and B
- without restrictions upon the delay of the discrete time model

The plant model should be stable and well damped !

Often used for the systems featuring a large delay

Remark: The name is misleading since it has nothing in common with the "internal model principle"

Regulation (computation of $R(q^{-1})$ and $S(q^{-1})$)

$$A(q^{-1})S(q^{-1}) + q^{-d}B(q^{-1})R(q^{-1}) = A(q^{-1})P_F(q^{-1}) = P(q^{-1}) \quad (*)$$

Dominant poles

$$P_F(q^{-1}) = \left(1 + \alpha q^{-1}\right)^{n_{P_F}}$$

(typical choice)

R should be in the form : $R(q^{-1}) = A(q^{-1}).R'(q^{-1})$

After the cancellation of the common factor $A(q^{-1})$,(*) becomes:

$$S(q^{-1}) + q^{-d}B(q^{-1})R'(q^{-1}) = P_F(q^{-1})$$

Solution for: $S(q^{-1}) = \left(1 - q^{-1}\right)S'(q^{-1})$ (typical choice)

$$R(q^{-1}) = A(q^{-1})\frac{P_F(1)}{B(1)}$$

$$S(q^{-1}) = (1 - q^{-1})S'(q^{-1}) = P_F(q^{-1}) - q^{-d}B(q^{-1})\frac{P_F(1)}{B(1)}$$

Tracking (computation of *T(q⁻¹)*)

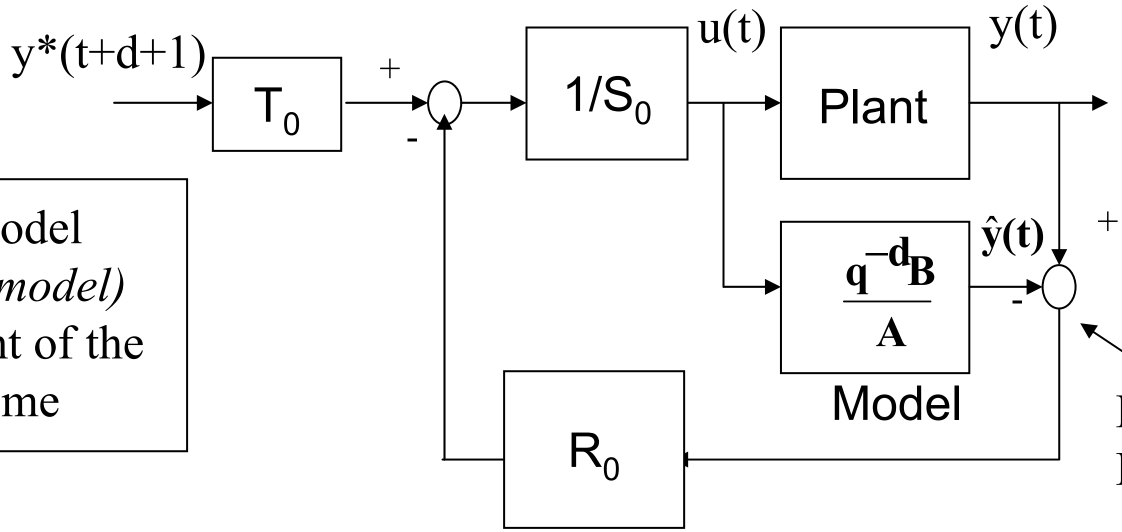
$$T(q^{-1}) = A(q^{-1})P_F(q^{-1}) / B(1)$$

Particular case : $A_m = AP_F$ (tracking dynamics = regulation dynamics)

$$T(q^{-1}) = T(1) = \frac{A(1)P_F(1)}{B(1)}$$ (cancellation of the tracking reference model)

Interpretation of the internal model control

Equivalent scheme



The plant model *(prediction model)* is an element of the control scheme

Feedback on the Prediction error

$$R_0(q^{-1}) = \frac{P_F(1)}{B(1)} A(q^{-1}) \;\; \left(for \; H_R(q^{-1}) = 1 \right)$$

$$S_0(q^{-1}) = P_F(q^{-1})$$

$$T_0(q^{-1}) = \frac{1}{B(1)} P(q^{-1}) = \frac{1}{B(1)} A(q^{-1}) P_F(q^{-1})$$

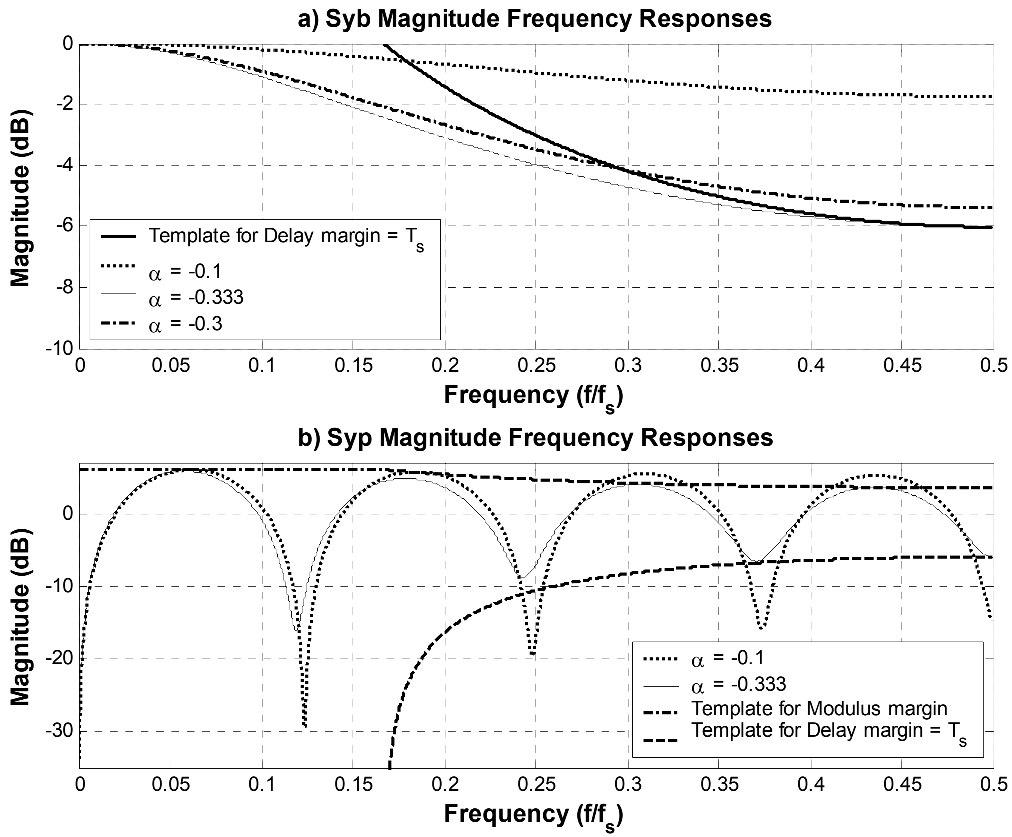
Rem.: *For all the strategies one can show the presence of the plant model in the controller*

Internal model control of a system with large delay

Plant: $d = 7;\ A = 1 - 0.2q^{-1}\ ;\ B = q^{-1}$

The « delay margin » can be satisfied by introducing auxiliary poles

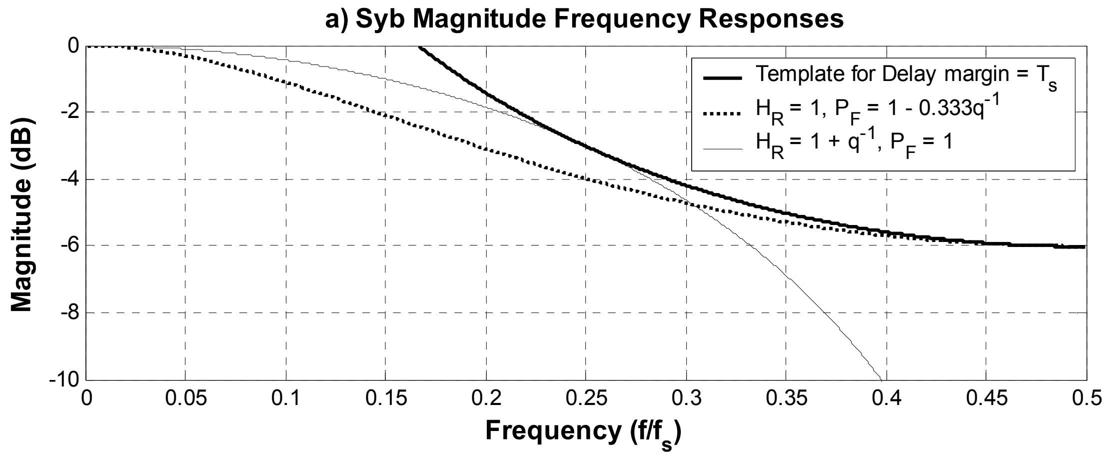
$$P_F(q^{-1}) = (1 + \alpha\, q^{-1}) \qquad -1 < \alpha < 0$$

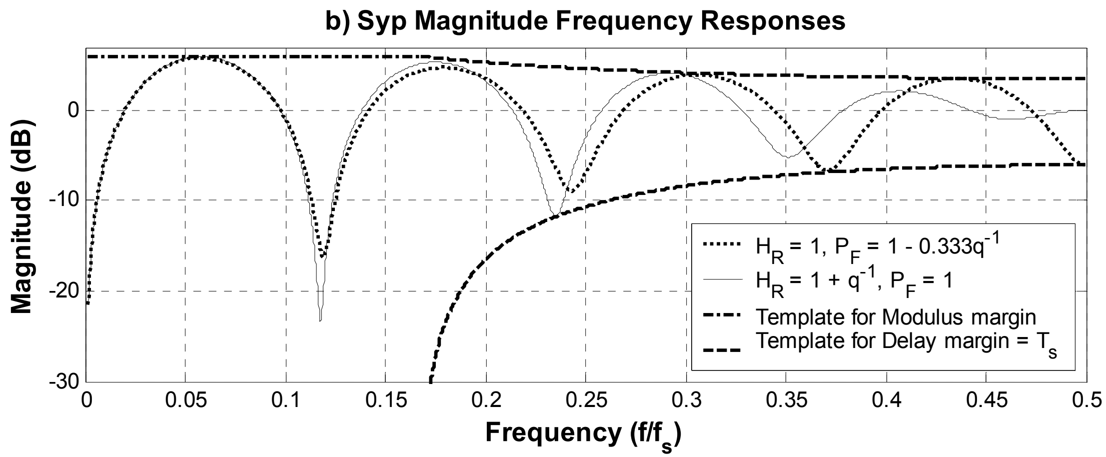


$\alpha = -0.1; -0.3; -0.333$

Good value

Internal model control of a system with large delay





$$H_R(q^{-1}) = 1 + q^{-1} \text{ corresponds to the opening of the loop at } 0.5f_S$$

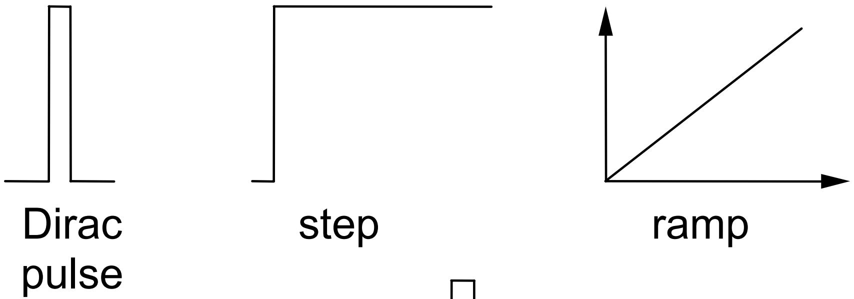
See also:

I.D. Landau (1995) : Robust digital control of systems with time delay (the Smith predictor revisited) Int. J. of Control, v.62,no.2 pp 325-347

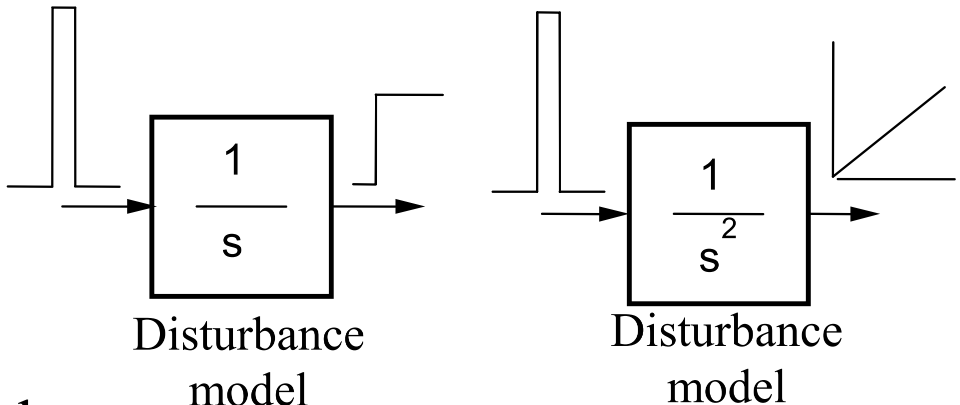
Minimum Variance Tracking and Regulation

Disturbance Representation

Deterministic disturbances



Can be described as a Dirac pulse passed through a filter



Stochastic Disturbances

Can not be described in a deterministic way, since they are not reproducible.

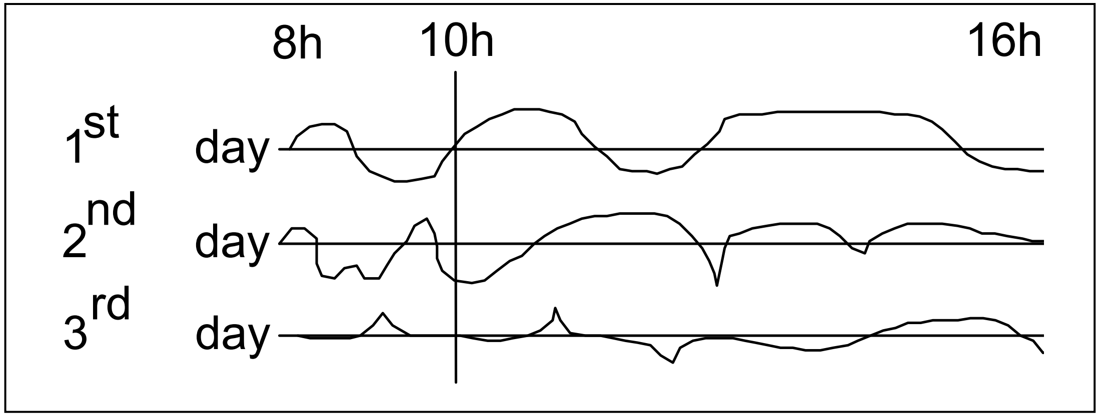
Most of the stochastic disturbances can be described as:

A white noise passed through a filter.

In a stochastic environment, the *white noise* play the role *of the Dirac Pulse*.

Stochastic (random) Process

Example: record of a controlled variable in regulation (1 day)

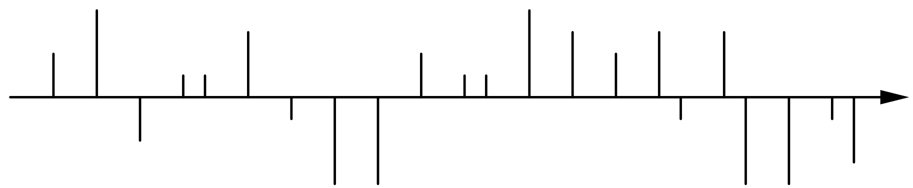


- each evolution can be described by a different *f(t)* (*stochastic realization*)
- for a fixed time (ex.: 10h) for each *experiment* (day) one gets a different measured value (*random variable*)
- one can define a *statistitics* (mean value, variance) and *probabilities* of occurrence of the various values
- if the *stochastic process* is *ergodic* the statistics over one *experiment* are significant
- if the *stochastic process* is *gaussian* the knowledge of the m.v. and variance allows to give the probability of occurrence of a certain value (Gauss bell – App.A)

Discrete-time Gaussian White Noise

It is the fundamental *generator signal*

$\{e(t)\}$: Sequence of independent equally distributed Gaussian random variables with rero mean and variance σ^2 $(0,\sigma\,)$ $\longleftarrow$ *standard deviation*



$$M.V. = E\{e(t)\} = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} e(t) = 0$$

$$var = E\{e^2(t)\} = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} e^2(t) = \sigma^2$$

Independence : The knowledge of *e(i)* does not allow to predict an approximation for *e(i+1), e(i+2)*....

Independence Test

Autocorrelation (covariance) function:

$$R(i) = E\{e(t)e(t-i)\} = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} e(t)e(t-i)$$

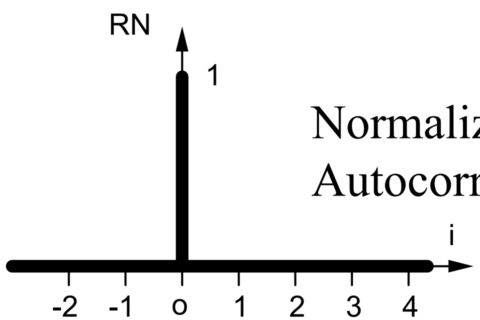
Rem.: R(0) = var = σ^2

Normalized autocorrelation (covariance) function :

$$RN(i) = \frac{R(i)}{R(0)} \qquad (RN(0) = 1)$$

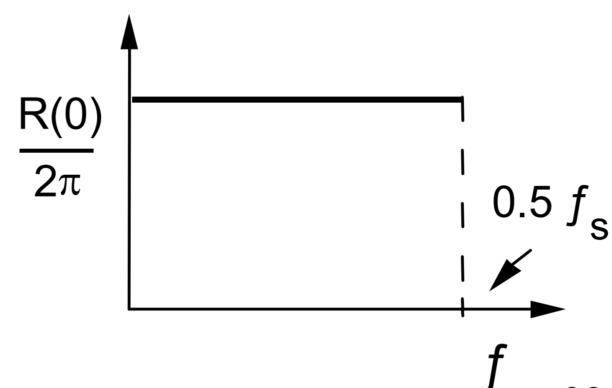
Whiteness (independence test) :$R(i)=RN(i) = 0 \quad i= 1, 2, 3..-1, -2 \ldots$

White Noise

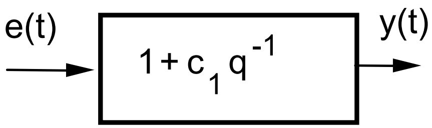


Normalized Autocorrelations

Spectral Density



Moving Average Process – MA



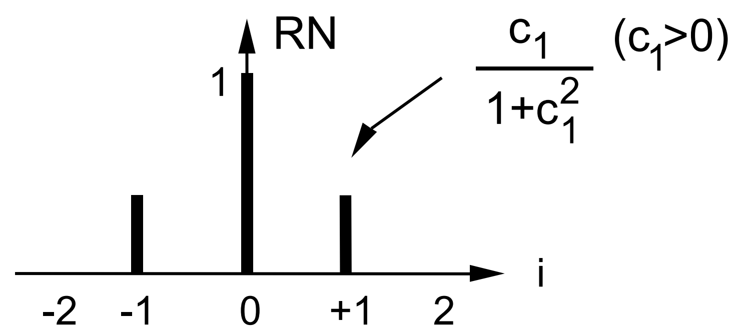
$$y(t) = e(t) + c_1 e(t-1) = (1 + c_1 q^{-1}) e(t)$$

$$V.M. = E\{y(t)\} = \frac{1}{N}\sum_{t=1}^{N} y(t) = \frac{1}{N}\sum_{t=1}^{N} e(t) + c_1 \frac{1}{N}\sum_{t=1}^{N} e(t-1) = 0$$

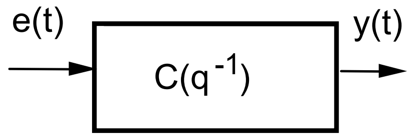
$$R_y(0) = E\{y^2(t)\} = \frac{1}{N}\sum_{t=1}^{N} y^2(t) = (1 + c_1^2)\sigma^2$$

$$R_y(1) = E\{y(t)y(t-1)\} = \frac{1}{N}\sum_{t=1}^{N} y(t)y(t-1) = \frac{1}{N} c_1 \sum_{t=1}^{N} e^2(t) = c_1^2 \sigma^2$$

$$R_y(2) = R_y(3) = .. = 0$$



Moving Average Process – MA



$$y(t) = e(t) + \sum_{i=1}^{n_c} c_i e(t-i) = C(q^{-1})e(t)$$

$$C(q^{-1}) = 1 + \sum_{i=1}^{n_c} c_i q^{-i} = 1 + q^{-1}C^*(q^{-1})$$

$$R(i) = 0 \quad i \geq n_C + 1 \quad i \geq -(n_C + 1)$$

Spectral density:

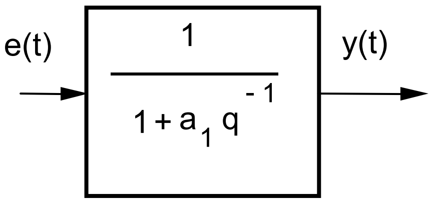
ϕ_e

$$\phi_y(\omega) = C(e^{j\omega})C(e^{-j\omega})\frac{\sigma^2}{2\pi} = \left|C(e^{j\omega})\right|^2 \frac{\sigma^2}{2\pi}$$

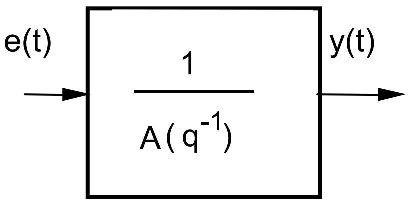
Relationship spectral density/transfer function:

$$\phi_y(z) = C(z)C(z^{-1})\phi_e(z)\,;\; z = e^{j\omega}$$

Auto-regressive Process – AR



$$y(t) = -a_1 y(t-1) + e(t) = \frac{e(t)}{1 + a_1 q^{-1}} \qquad |a_1| < 1$$



$$y(t) = -\sum_{i=1}^{n_A} a_i y(t-1) + e(t) \longrightarrow A(q^{-1})y(t) = e(t)$$

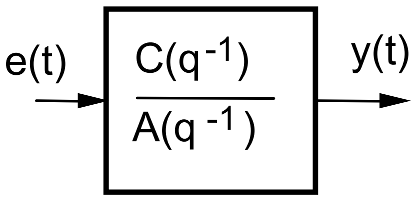
$$A(q^{-1}) = 1 + \sum_{i=1}^{n_A} a_i q^{-i} = 1 + q^{-1} A^*(q^{-1})$$

Asymptotically stable

Spectral density:

$$\phi_y(z) = \frac{1}{A(z)} \frac{1}{A(z^{-1})} \phi_e(z) \qquad \phi_y(\omega) = \phi_y(z)\Big|_{z=e^{j\omega}}$$

Auto-regressive Moving Average Process - ARMA



$$y(t) = -\sum_{i=1}^{n_A} a_i y(t-i) + \sum_{i=1}^{n_C} c_i e(t-i) + e(t)$$

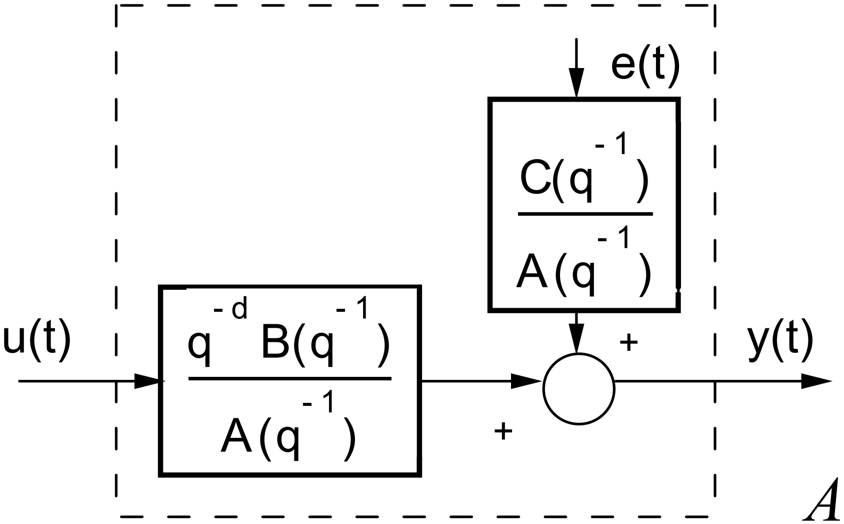
$$A(q^{-1})y(t) = C(q^{-1})e(t)$$

Asymptotically stable

Spectral density:

$$\phi_y(z) = \left(\frac{C(z)}{A(z)}\right)\left(\frac{C(z^{-1})}{A(z^{-1})}\right)\phi_e(z)$$

ARMAX Process (*A.R.M.A. with « exogenous » input)*)



$$y(t) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})}u(t) + \boxed{\frac{C(q^{-1})}{A(q^{-1})}e(t)}$$

Disturbance

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + C(q^{-1})e(t)$$

$$y(t+1) = -\sum_{i=1}^{n_A} a_i y(t+1-i) + \sum_{i=1}^{n_B} b_i u(t+1-d-i) + \sum_{i=1}^{n_C} c_i e(t+1-i) + e(t+1)$$

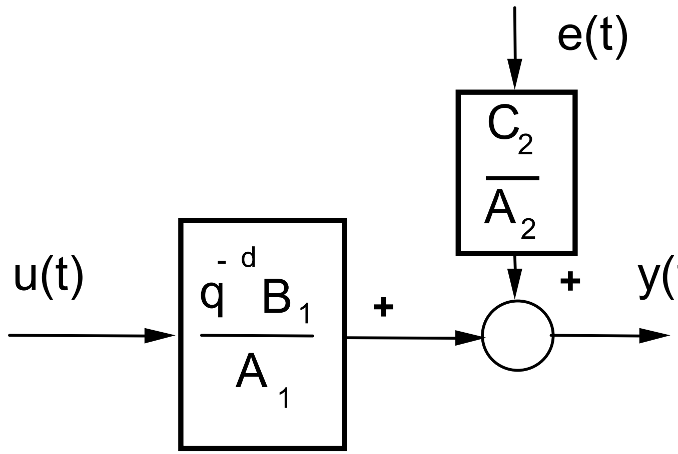
$$y(t+1) = -A^*(q^{-1})y(t) + B^*(q^{-1})u(t-d) + C(q^{-1})e(t+1)$$

Remark: in general $n_C = n_A$

Example: $n_A = 1\,;\, n_B = 1\,;\, n_C = 1\,;\, d = 0$

$$y(t+1) = -a_1 y(t) + b_1 u(t) + c_1 e(t) + e(t+1)$$

Generality of the ARMAX Process



$$y(t)=\frac{q^{-d}B_1(q^{-1})}{A_1(q^{-1})}u(t)+\frac{C_2(q^{-1})}{A_2(q^{-1})}e(t)$$

$$\Downarrow$$

$$y(t)=\frac{q^{-d}B_1A_2}{A_1A_2}u(t)+\frac{C_2A_1}{A_1A_2}e(t)=\frac{q^{-d}B}{A}u(t)+\frac{C}{A}e(t)$$

$$A=A_1A_2\ ;\ B=B_1A_2\ ;\ C=C_2A_1$$

Optimal Prediction

$\hat{y}(t+1/t) =$ Prediction of *y(t+1)* based on the measures of *u* and *y* available up to *t*

Prediction error: $\varepsilon(t+1) = y(t+1) - \hat{y}(t+1)$

Objective: $\hat{y}(t+1/t) = \hat{y}(t+1) = f(y(t), y(t-1), \ldots, u(t), u(t-1), \ldots)$

such that : $E\left\{[y(t+1) - \hat{y}(t+1)]^2\right\} = \min$

Example : $y(t+1) = -a_1 y(t) + b_1 u(t) + c_1 e(t) + e(t+1)$

$$\varepsilon(t+1) = y(t+1) - \hat{y}(t+1) = [-a_1 y(t) + b_1 u(t) + c_1 e(t) - \hat{y}(t+1)] + e(t+1)$$

$$E\left\{[y(t+1) - \hat{y}(t+1)]^2\right\} = E\left\{[-a_1 y(t) + b_1 u(t) + c_1 e(t) - \hat{y}(t+1)]^2\right\} + E\left\{e^2(t+1)\right\}$$
$$+ 2\underbrace{E\{e(t+1)[-a_1 y(t) + b_1 u(t) + c_1 e(t)]\}}_{=0}$$

Optimality condition: $E\left\{[-a_1 y(t) + b_1 u(t) + c_1 e(t) - \hat{y}(t+1)]^2\right\} = 0$

$$\hat{y}(t+1)\Big|_{opt} = -a_1 y(t) + b_1 u(t) + c_1 e(t) \longrightarrow \varepsilon(t+1)\Big|_{opt} = y(t+1) - \hat{y}(t+1)\Big|_{opt} = e(t+1)$$

$$\varepsilon(t) = e(t) \longrightarrow \hat{y}(t+1)\Big|_{opt} = -a_1 y(t) + b_1 u(t) + c_1 \varepsilon(t)$$

Optimal prediction

ARMAX:

$$y(t+1) = -A^*(q^{-1})y(t) + B^*(q^{-1})u(t-d) + C(q^{-1})e(t+1)$$

Optimal predictor (theoretical):

$$\hat{y}(t+1) = -A^*(q^{-1})y(t) + B^*(q^{-1})u(t-d) + C^*(q^{-1})e(t)$$

Prediction error:

$$\varepsilon(t+1)\big|_{opt} = y(t+1) - \hat{y}(t+1) = e(t+1)$$

Optimal predictor (implementation):

$$\hat{y}(t+1) = -A^*(q^{-1})y(t) + B^*(q^{-1})u(t-d) + C^*(q^{-1})\varepsilon(t)$$

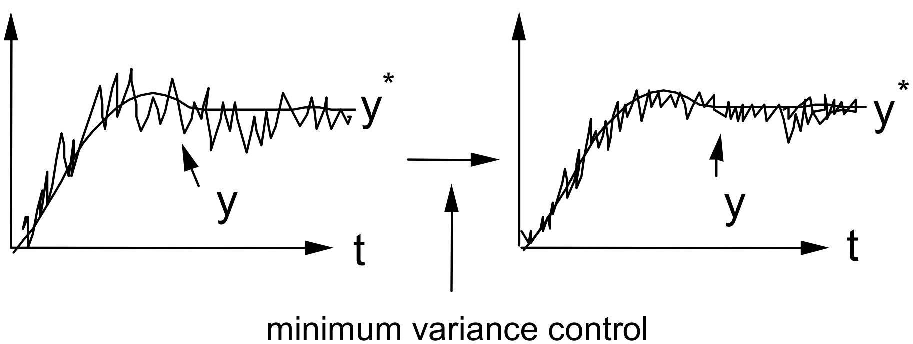
One replaces the unknown white noise by the prediction error

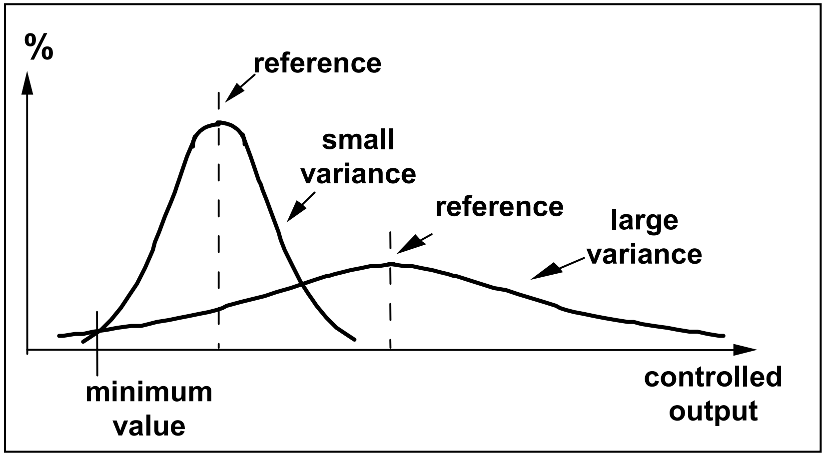
Minimum Variance Tracking and Regulation

- random disturbances
- the discrete time plant model has stable zeros

Objective: *minimization of the output variance (standard deviation)*

$$J(u(t)) = E\left\{\left[y(t) - y^*(t)\right]^2\right\} \approx \frac{1}{N}\sum_{t=1}^{N}\left[y(t) - y^*(t)\right]^2 = \min$$





- A model for the disturbance has to be considered
- Plant + disturbance: ARMAX model

Minimum Variance Tracking and Regulation

Plant + disturbance: $y(t+1) = - a_1 y(t) + b_1 u(t) + b_2 u(t-1) + c_1 e(t) + e(t+1)$

Reference trajectory: $y^*(t+1)$

Criterion computation:

$$E\{[y(t+1) - y^*(t+1)]^2\} = E\{[-a_1 y(t) + b_1 u(t) + b_2 u(t-1) + c_1 e(t) - y^*(t+1)]^2\}$$
$$+E\{e^2(t+1)\} + \underbrace{2E\{e(t+1)[- a_1 y(t) + b_1 u(t) + b_2 u(t-1) + c_1 e(t) - y^*(t+1)]\}}_{=0}$$

Optimality condition: $E\{[-a_1 y(t) + b_1 u(t) + b_2 u(t-1) + c_1 e(t) - y^*(t+1)]^2\} = 0$

Control law (theoretical):

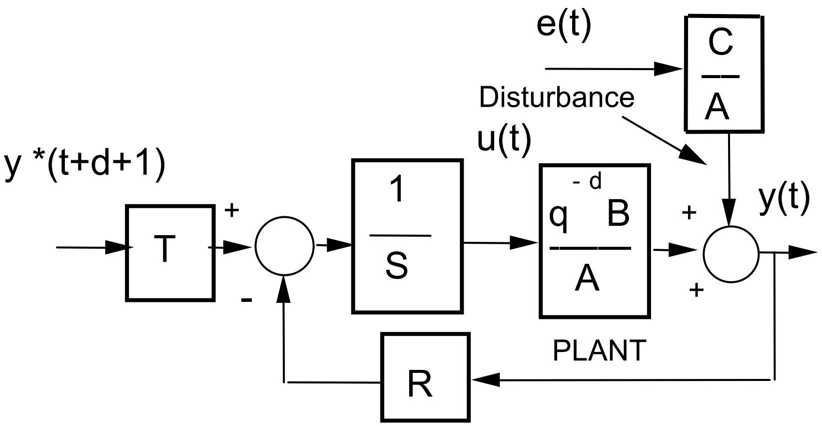
$$u(t) = \frac{y^*(t+1) - c_1 e(t) + a_1 y(t)}{b_1 + b_2 q^{-1}} \rightarrow y(t+1) - y^*(t+1) = e(t+1) \longrightarrow y(t) - y^*(t) = e(t)$$

Control law (implementation):

$$u(t) = \frac{(1 + c_1 q^{-1}) y^*(t+1) - (c_1 - a_1) y(t)}{b_1 + b_2 q^{-1}} = \frac{T(q^{-1}) y^*(t+1) - R(q^{-1}) y(t)}{S(q^{-1})}$$

Same control law as for « Tracking and regulation with independent objectives » by taking $P(q^{-1}) = C(q^{-1})$

Closed Loop Poles



$$u(t) = \frac{T(q^{-1})y^*(t+1) - R(q^{-1})y(t)}{S(q^{-1})}$$

$$H_{BF}(q^{-1}) = \frac{T(q^{-1})q^{-(d+1)}B^*(q^{-1})}{A(q^{-1})S(q^{-1}) + q^{-1}B^*(q^{-1})R(q^{-1})}$$

$A(q\text{-}1) = 1 + a_1 q^{-1}$; $B(q^{-1}) = q^{-1} B^*(q^{-1})$; $B^*(q^{-1}) = b_1 + b_2 q^{-1}$; $d = 0$

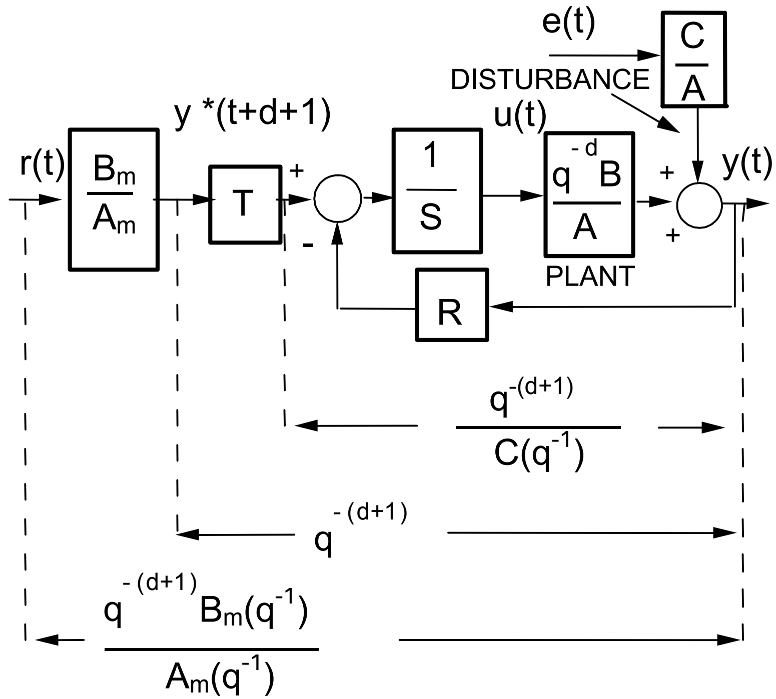
$T(q^{-1}) = C(q^{-1}) = 1 + c_1 q^{-1}$; $S(q^{-1}) = B^*(q^{-1}) = b_1 + b_2 q^{-1}$; $R(q^{-1}) = r_0 = c_1 - a_1$

$$H_{BF}(q^{-1}) = \frac{T(q^{-1})q^{-1}}{A(q^{-1}) + q^{-1}R(q^{-1})} = \frac{T(q^{-1})q^{-1}}{C(q^{-1})} = q^{-1}$$

Closed loop poles

The disturbance model ($C(q^{-1})$) defines the closed loop poles and therefore the regulation performance

Minimum Variance Tracking and Regulation – general case



Same computations as for « tracking and regulation with independent objectives » by taking $P(q^{-1}) = C(q^{-1})$ (see Chapter 3)

Minimum Variance Tracking and Regulation – General Case

$$u(t) = \frac{T(q^{-1})y^*(t+d+1) - R(q^{-1})y(t)}{S(q^{-1})}$$

$$T(q^{-1}) = C(q^{-1})\;;\; S(q^{-1}) = B^*(q^{-1})S'(q^{-1})$$

$$A(q^{-1})S'(q^{-1}) + q^{-(d+1)}B^*(q^{-1})R(q^{-1}) = C(q^{-1})$$

Solving with *predisol.sci(.m) or with WinReg (Adaptech)*

Prediction error : $$y(t+d+1) - y^*(t+d+1) = \underbrace{S'(q^{-1})e(t+d+1)}_{\text{MA of order } d}$$

Optimality test:

$$R(i) = \frac{1}{N}\sum_{t=1}^{N}\left[y(t) - y^*(t)\right]\cdot\left[y(t-i) - y^*(t-i)\right] \qquad i = 0,1,2,\ldots$$

$$\boxed{RN(i) \approx 0 \quad i \geq d+1}$$

theoretical

$$\boxed{|RN(i)| \leq 02.17\sqrt{N} \quad i \geq d+1}$$

pratical

Minimum Variance Tracking and Regulation. Example

Plant:

- *d = 0*
- *B(q-1) = 0.2 q-1 + 0.1 q-2*
- *A(q-1) = 1 - 1.3 q-1 + 0.42 q-2*

Tracking dynamics → *Ts = 1s, ω0 = 0.5 rad/s, ζ = 0.9*

- *Bm = +0.0927 +0.0687 q-1*
- *Am = 1 - 1.2451 q-1 + 0.4066 q-2*

Disturbance polynomial → C(q-1) = 1 -1.34 q-1 + 0.49 q-2

Pre-specifications: Integrator

****** CONTROL LAW ******

S(q-1) u(t) + R(q-1) y(t) = T(q-1) y(t+d+1)*

y(t+d+1) = [(Bmq-1)/Am(q-1)] . ref(t)*

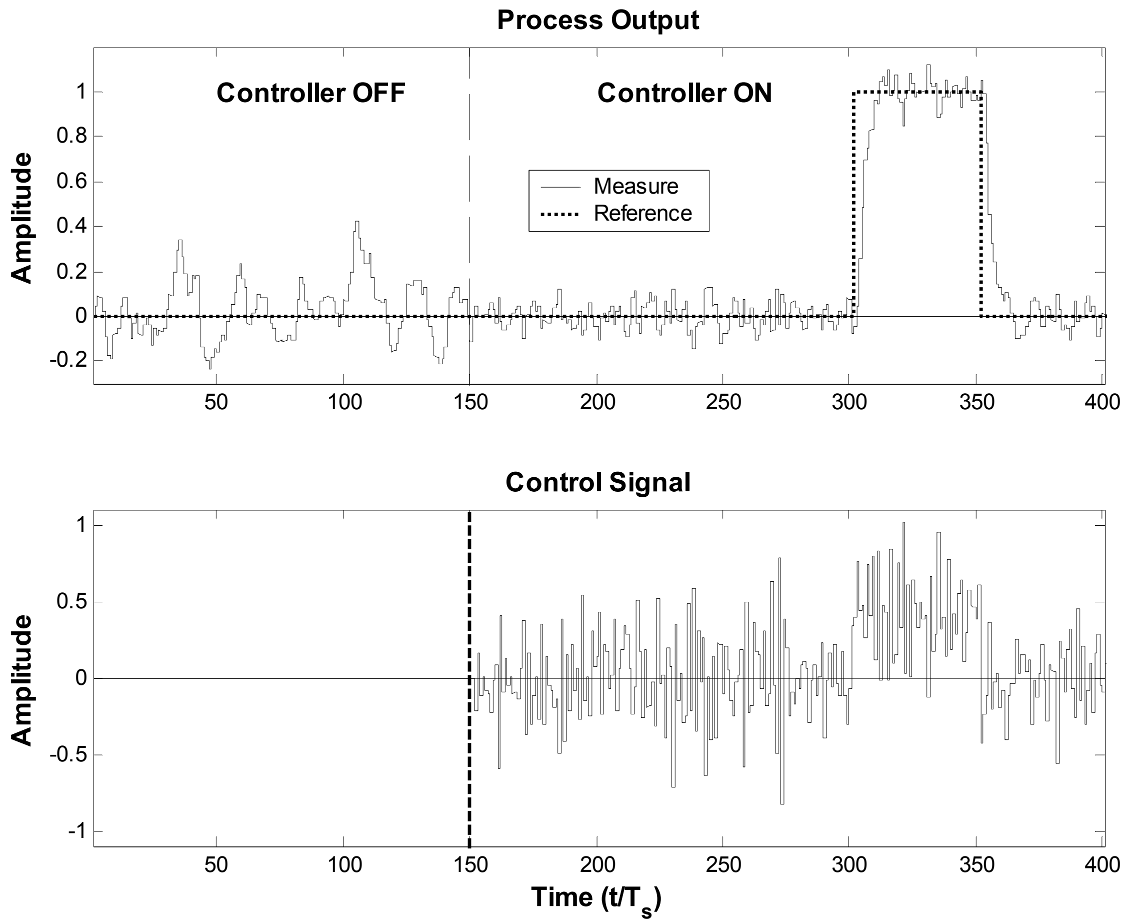
Controller:

- *R(q-1) = 0.96 - 1.23 q-1 + 0.42 q-2*
- *S(q-1) = 0.2 - 0.1 q-1 - 0.1 q-2*
- *T(q-1) = C(q-1)*

Gain margin: 2.084 *Phase margin: 61.8 deg*

Modulus margin: 0.520 (- 5.68 dB) *Delay margin: 1.3 s*

Minimum Variance Tracking and Regulation. Example



Attention: For robustness and actuator stress one may be obliged to add auxiliary poles (see book pg. 190)

Minimum Variance Tracking and Regulation
The case of unstable zeros

In this case minimum variance control can not be applied

Solutions:

- Use of pole placement with a special choice of the closed loop poles
- Generalized minimum variance tracking and regulation (modified criterion)

Use of pole placement

$$B^*(q^{-1}) = B^+(q^{-1})B^-(q^{-1})$$

Unstable factor

$B^{-}{}'(q^{-1})$ Reciprocical polynomial (stable) of $B^-(q^{-1})$

(one reverses the order of the coefficients)

Closed loop poles

$$P(q^{-1}) = B^+(q^{-1})B^{-}{}'(q^{-1})C(q^{-1})$$
$$= A(q^{-1})S(q^{-1}) + q^{-(d+1)}B^*(q^{-1})R(q^{-1})$$

For details and examples, see book pg.192-195

Generalized Minimum Variance Tracking and Regulation

Criterion:

$$E\left\{\left[y(t+d+1)-y^*(t+d+1)+\frac{Q(q^{-1})}{C(q^{-1})}u(t)\right]^2\right\}=\min$$

$$Q(q^{-1})=\frac{\lambda(1-q^{-1})}{1+\alpha q^{-1}}$$

Particular case : $\alpha = 0$

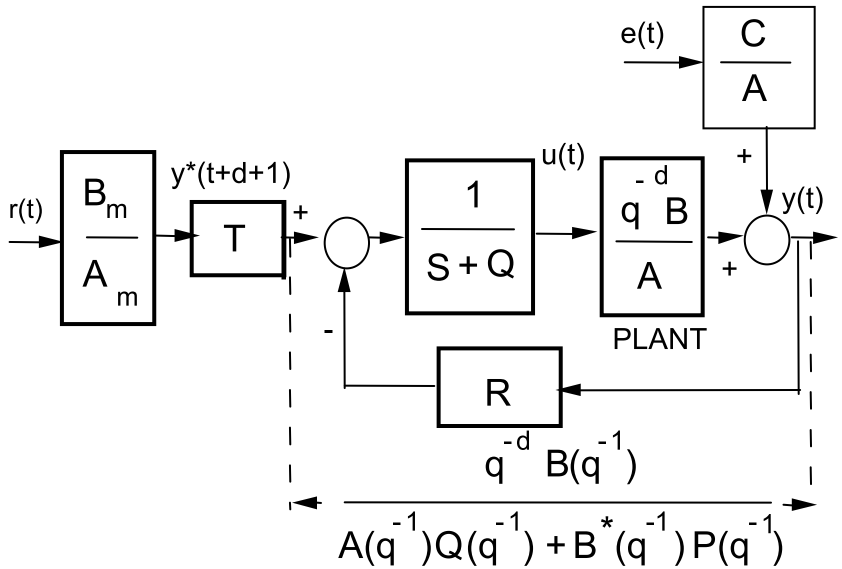
Weighting the control variations

$$E\left\{\left[y(t+d+1)+\frac{\lambda}{C(q^{-1})}[u(t)-u(t-1)]-y^*(t+d+1)\right]^2\right\}=\min$$

Controllere: $$u(t)=\frac{C(q^{-1})y^*(t+d+1)-R(q^{-1})y(t)}{S(q^{-1})+Q(q^{-1})}$$

Allows to stabilize the controller and the system (but not always!)

Generalized minimum variance tracking and regulation



Design:

- One computes a minimum variance tracking/regulation controller without taking in account the unstable nature of *B*. *($Q(q^{-1})=0$)*
- One introduces *$Q(q^{-1})$* and search for $\lambda > 0$ wich stabilizes the controller and the closed loop

A solution does not always exists in particular when there are several unstable zeros